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CENTER FOR STOCHASTIC PROCESSES

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ON EXCEEDANCE POINT PROCESSES FOR STATIONARY SEQUENCES
UNDER MILD OSCILLATION RESTRICTIONS

by

M.R. Leadbetter

and

S. Nandagopalan

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ON EXCEEDANCE POINT PROCESSES FOR STATIONARY SEQUENCES UNDER MILD OSCILLATION RESTRICTIONS

by

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and

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Summary:

It is known ([1]) that any point process limits for the (time normalized) exceedances of high levels by a stationary sequence is necessarily Compound Poisson, under general dependence restrictions. This results from the clustering of exceedances where the underlying Poisson points represent cluster positions, and the multiplicities correspond to cluster sizes.

Here we investigate a class of stationary sequences satisfying a mild local dependence condition restricting the extent of local "rapid oscillation". For this class, criteria are given for the existence and value of the so-called "extremal index" which plays a key role in determining the intensity of cluster positions. Cluster size distributions are investigated for this class and in particular shown to be asymptotically equivalent to those for lengths of runs of consecutive exceedances above the level. Relations between the point processes of exceedances, cluster centers, and upcrossings are discussed.

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1. Introduction and basic results.

The paper [1], provides limiting results for the (time normalized) point process N_n of exceedances of a high level u_n by a stationary sequence $\{\xi_n\}$. It is shown, for example, that typically a limit for N_n must have a Compound Poisson form where the underlying Poisson points may be regarded as positions of "clusters" of exceedances and the multiplicities correspond to cluster sizes, i.e. the number of exceedances in a cluster.

Let $\xi_1, \xi_2 \dots$ be a stationary sequence. Write $M_n = \max(\xi_1, \xi_2, \dots, \xi_n)$ and for $\tau > 0$ let $u_n(\tau)$ denote levels such that $n(1-F(u_n(\tau))) \to \tau$, where F is the distribution function (d.f.) of each ξ_i. Then it is often the case that $P\{M_n \le u_n(\tau)\} \to e^{-\theta\tau}$ where θ is a fixed parameter $(0 \le \theta \le 1)$, referred to as the "extremal index" of the sequence. It is known that $\theta = 1$ for i.i.d. sequences and many dependent cases, and that $\theta > 0$ for "almost all" cases of interest. For such levels $u_n(\tau)$ it may be shown under general conditions that the intensity for the Poisson limiting cluster positions in N_n is simply $\theta\tau$.

These results require a restriction on the long range dependence of the sequence, and two such conditions ($D(u_n)$, $\Delta(u_n)$, defined below) are useful. It is well known that under a further short range dependence condition ($D'(u_n)$ - cf. [3] Section 3.4) it may be shown that $\theta = 1$ and the Compound Poisson limit for N_n becomes Poisson. In this paper we consider a special but much wider class of sequences subject to a weaker condition which restricts rapid oscillations - here called $D''(u_n)$ - than $D'(u_n)$, and for which all values of θ in (0,1] are possible. It will be shown for this class that the joint distribution of ξ_1 and ξ_2 determines whether the extremal index exists, and gives its value. Finally for this class clusters of exceedances may be simply identified asymptotically as runs of consecutive exceedances and the cluster sizes as run lengths.

Section 2 contains the theory surrounding the maximum and the extremal index when the local dependence condition $D''(u_n)$ holds, and in Section 3 asymptotic properties of point processes of exceedances, upcrossings and cluster centers are discussed. Notation used throughout will include $M(E)$ to denote $\max\{\xi_i: i \in E\}$ for any set $E \subset (0,n]$ ($M_n = M[1,n]$). A time scale normalization by $1/n$ will be used to define various point processes on the unit interval. In particular the exceedance point process N_n is defined with respect to a sequence of "levels" $\{u_n\}$ by

(1.1) $$N_n(B) = \#\{i,\ 1 \leq i \leq n:\ i/n \in B,\ \xi_i > u_n\}$$

for each Borel subset B of (0,1]. This involves a slight awkwardness of notation in that $M(E)$ is defined for subsets E of (0,n], whereas $N_n(B)$ is defined for $B \subset (0,1]$ when writing an equivalence $\{N_n(B) = 0\} = \{M(nB) \leq u_n\}$ but a more intricate notation does not seem worthwhile.

The long range dependence condition $D(u_n)$ is defined as follows. Abbreviate $F_{i_1 \ldots i_n}(u,u \ldots u)$ to $F_{i_1 \ldots i_n}(u)$. Then for a sequence $\{u_n\}$, $D(u_n)$ is said to hold if for each n, $1 \leq i_1 < i_2 \ldots < i_p < j_1 \ldots < j_{p'} \leq n$, $j_1 - i_p \geq \ell$ we have

$$|F_{i_1 \ldots i_p j_1 \ldots j_{p'}}(u_n) - F_{i_1 \ldots i_p}(u_n) F_{j_1 \ldots j_{p'}}(u_n)| \leq \alpha_{n,\ell}$$

where $\alpha_{n,\ell_n} \to 0$ for some $\ell_n = o(n)$. Frequently integers $k_n \to \infty$ will be chosen so that

(1.2) $$k_n \alpha_{n,\ell_n} \to 0, \qquad k_n \ell_n / n \to 0$$

Note that, by $D(u_n)$, this holds automatically for bounded k_n-sequences but $k_n \to \infty$ can clearly be chosen so that (1.2) is satisfied. Note also that the condition $D(u_n)$ is of similar type to (but much weaker than) strong mixing. In the following basic result and throughout, m will denote Lebesgue measure. The result is a slightly more general form of Lemma 2.3 of [1].

<u>Lemma 1.1</u> *Let* $D(u_n)$ *hold and* $\{k_n\}$ *satisfy* (1.2). Let J_i $(=J_{i,n})$, $1 \le i \le k_n$, be disjoint subintervals of (0,1] with $\frac{n}{k_n \ell_n} \sum_1^{k_n} m(J_i) \to \infty$ (which holds, in particular, if $m(\bigcup_1^{k_n} J_i) \to \alpha > 0$). Then

(i) $\gamma_n = P\{M(\bigcup_1^{k_n} nJ_i) \le u_n\} - \prod_1^{k_n} P\{M(nJ_i) \le u_n\} \to 0$ as $n \to \infty$

(ii) If J is a fixed subinterval of (0,1] with $m(J) = \alpha$, $\bigcup_1^{k_n} J_i \subset J$ and $m(\bigcup_1^{k_n} J_i) \to \alpha$, then

$$P\{M(nJ) \le u_n\} - \prod_1^{k_n} P\{M(nJ_i) \le u_n\} \to 0. \tag{1.3}$$

Proof: The assertion (i) is proved by arguments very close to those used in Lemma 2.2 of [1]. The main difference is the complicating feature in that here we do not assume that $m(J_i) \ge \ell_n/n$ for each i, but clearly the intervals J_i for which $m(J_i) < \ell_n/n$ form a set whose total measure cannot exceed $k_n\ell_n/n \to 0$. The proof of (i) will not be given in detail, though its flavor may be seen from the sketch for (ii) below. It is in fact very simple in the usual situation where exceedances in short intervals are unlikely in the sense that $k_n P\{M_{\ell_n} > u_n\} \to 0$, and is made more lengthy to cover cases when this does not hold by *showing that both terms of* (i) actually tend to zero.

(ii) (sketch of proof). By stationarity the intervals J_i may be taken to be abutting and $\bigcup_1^{k_n} J_i = I_n$, $J - I_n = I_n^*$ taken to be intervals without affecting either term of (1.3), and $m(I_n^*) \to 0$. By (i), (ii) will follow if

$$\gamma_n' = P\{M(nI_n) \le u_n\} - P\{M(nJ) \le u_n\} \to 0$$

and it is sufficient to show that if γ_n' has a limit as $n \to \infty$ through a subsequence S, then that limit is zero. This is immediate if $P\{M(nI_n^*) > u_n\}$ tends to zero, since this probability dominates γ_n'. Otherwise $P\{M(nI_n^*) > u_n\} \to \alpha > 0$ as $n \to \infty$ through some subsequence $S' \subset S$. Clearly $\theta_n(\to \infty)$ copies $I_{n,j}$ of I_n^*, each separated by at least ℓ_n/n, may be placed in I_n, and $P\{M(nI_n) \le u_n\}$ thus dominated by $P\{\bigcap_1^{\theta_n}(M(nI_{n,j}) \le u_n)\}$. By appropriate choice of θ_n this probability may be approximated by $P^{\theta_n}\{M(nI_n) \le u_n\}$ (using $D(u_n)$) which tends to zero as $n \to \infty$ through S'. Hence the first term in γ_n' tends to zero and it dominates the second, which thus also tends to zero. □

2. Extremal theory under $D''(u_n)$.

If $D(u_n)$ holds, and k_n are integers satisfying (1.2), and $k_n(1-F(u_n)) \to 0$, $r_n = [n/k_n]$, define

$$D''(u_n):\ n \sum_{j=2}^{r_n-1} P\{\xi_1 > u,\ \xi_j \le u_n < \xi_{j+1}\} \to 0.$$

and write $\mu(u) = P\{\xi_1 \le u < \xi_2\}$. We say that $\{\xi_n\}$ has an upcrossing of u at j if $\xi_{j-1} \le u < \xi_j$, so that $\mu(u)$ may obviously be interpreted as the mean number of upcrossings of u per unit time. (This notation will be used throughout this and the next section without comment). The condition $D''(u_n)$ involves a weaker restriction than $D'(u_n)$ of [3] which is used to guarantee that $\theta = 1$, whereas under D" all values $0 \le \theta \le 1$ are possible. For most of our purposes D" can be slightly weakened by replacing "$\xi_1 > u_n$" by "$\xi_1 \le u_n < \xi_2$" thus restricting the local occurrence of two or more upcrossings, but the present form is convenient for use here.

Proposition 2.1. Suppose $D(u_n)$, $D''(u_n)$ hold for given constants $\{u_n\}$, $\{k_n\}$, $\{r_n\}$ as above and write

$$\nu = \liminf n\mu(u_n), \quad \nu' = \limsup n\mu(u_n).$$

Then

$$\liminf P\{M_n \le u_n\} = e^{-\nu'}, \quad \limsup P\{M_n \le u_n\} = e^{-\nu}.$$

In particular $P\{M_n \le u_n\} \to e^{-\nu}$ if and only if $n\mu(u_n) \to \nu$.

Proof: Write $A_j = \{\xi_j \le u_n < \xi_{j+1}\}$. Then $\{M_{r_n} > u_n\} = \{\xi_1 > u_n\} \cup \bigcup_{j=1}^{r_n-1} A_j$ so that

$$\sum_{j=1}^{r_n-1} P(A_j) - \sum_{1 \le i < j \le r_n-1} P(A_i \cap A_j) \le P\{M_{r_n} > u_n\} \le 1-F(u_n) + \sum_{j=1}^{r_n-1} P(A_j)$$

Hence, since $P(A_j) = \mu(u_n)$, (and using stationarity),

$$(r_n-1)\mu(u_n) - S_n \le P\{M_{r_n} > u_n\} \le 1-F(u_n) + (r_n-1)\mu(u_n)$$

in which $S_n = r_n \sum_{j=2}^{r_n-1} P\{\xi_1 > u_n, \ \xi_j \le u_n < \xi_{j+1}\} = o(k_n^{-1})$ by $D''(u_n)$.
Multiplication by k_n yields

$$n\mu(u_n)(1 + o(1)) - o(1) \le k_n P\{M_{r_n} > u_n\} \le n\mu(u_n) + o(1).$$

From which it follows that

$$\limsup k_n P\{M_{r_n} > u_n\} = \nu', \quad \liminf k_n \, P\{M(r_n) > u_n\} = \nu.$$

Now by Lemma 1.1,

$$P\{M_n \le u_n\} = \left(1 - \frac{k_n P\{M_{r_n} > u_n\}}{k_n}\right)^{k_n} + o(1).$$

For $\epsilon > 0$, $k_n P\{M_{r_n} > u_n\} \ge \upsilon - \epsilon$ for sufficiently large n, so that $P\{M_n \le u_n\} \le (1 - \frac{\upsilon-\epsilon}{k_n})^{k_n} + o(1) \to e^{-\upsilon+\epsilon}$ and hence $\limsup P\{M_n \le u_n\} \le e^{-\upsilon}$. Similarly $P\{M_n \le u_n\} \ge (1 - \frac{\upsilon+\epsilon}{k_n})^{k_n} + o(1)$ for infinitely many values of n so that $\limsup P\{M_n \le u_n\} \ge e^{-\upsilon-\epsilon}$ and hence $\limsup P\{M_n \le u_n\} \ge e^{-\upsilon}$, showing that $\limsup P\{M_n \le u_n\} = e^{-\upsilon}$. Similarly $\liminf P\{M_n \le u_n\} = e^{-\upsilon}$, as required. □

<u>Corollary 2.2</u> If $I_j = (a_j, b_j]$ are disjoint subintervals of $(0,1]$, $1 \le j \le k$, then under the conditions of Proposition 2.1, if $n\mu(u_n) \to \upsilon$,

$$P\{\bigcap_1^k (M(nI_j) \le u_n)\} \to \exp\{-\upsilon \sum_1^k (b_j - a_j)\}$$

Proof: It follows from Lemma 1.1 that $P\{\bigcap_1^k (M(nI_j) \le u_n)\} - \prod_1^k P\{M(nI_j) \le u_n\} \to 0$ so that it is only necessary to show the result for k=1. Let k_n be as in Proposition 2.1, $r_n = [n/k_n]$. Then it follows readily from Lemma 1.1 and Proposition 2.1 that

$$P^{k_n}\{M_{r_n} \le u_n\} = P\{M_n \le u_n\} + o(1) \to e^{-\upsilon}$$

and hence that for $0 < a < b \le 1$,

$$\begin{aligned} P\{M((na,nb]) \le u_n\} &= (P\{M_{r_n} \le u_n\})^{([nb]-[na])/r_n} \\ &= (P\{M_{r_n} \le u_n\})^{k_n(b-a)(1+o(1))} \\ &\to e^{-\upsilon(b-a)} \end{aligned}$$

as required to complete the proof. □

We consider now levels $u_n = u_n(\tau)$ defined to satisfy $n(1-F(u_n(\tau))) \to \tau$. Note first the simply proved relation

$$\text{(2.1)} \qquad \mu(u) = P\{\xi_1 \le u < \xi_2\} = P\{\xi_2 \le u | \xi_1 > u\}\ (1-F(u))$$

Proposition 2.1 may be applied as follows.

<u>Proposition 2.3</u> Assume $D(u_n)$, $D''(u_n)$ hold for $u_n = u_n(\tau)$, some $\tau > 0$. Write $\theta = \liminf P\{\xi_2 \le u_n(\tau) | \xi_1 > u_n(\tau)\}$, $\theta' = \limsup P\{\xi_2 \le u_n(\tau) | \xi_1 > u_n(\tau)\}$. Then $\limsup P\{M_n \le u_n(\tau)\} = e^{-\theta\tau}$, $\liminf P\{M_n \le u_n(\tau)\} = e^{-\theta'\tau}$

Proof: By (2.1),

$$\nu = \liminf n\mu(u_n(\tau)) = \theta\tau, \quad \nu' = \limsup n\mu(u_n(\tau)) = \theta'\tau$$

and the results follow at once from Proposition 2.1. □

If $P\{M_n \le u_n(\tau)\} \to e^{-\theta\tau}$ for all $\tau > 0$ the parameter θ will be referred to as the *extremal index* of the sequence $\{\xi_n\}$. It is known (cf. [3] Theorem 3.7.1) that if $D(u_n(\tau))$ holds for each $\tau > 0$ and $P\{M_n \le u_n(\tau)\}$ converges for some $\tau > 0$, then $P\{M_n \le u_n(\tau)\}$ converges for all $\tau > 0$ and the limit has the form $e^{-\theta\tau}$ for fixed θ, $0 \le \theta \le 1$, i.e. the extremal index then exists. The following result, gives a convenient existence criterion assuming also $D''(u_n)$, and follows immediately from Proposition 2.3 and these observations.

<u>Corollary 2.4</u> Assume $D(u_n(\tau))$, $D''(u_n(\tau))$ hold for each $\tau > 0$. If $P\{\xi_2 \le u_n(\tau) | \xi_1 > u_n(\tau)\} \to \theta$ for some $\tau > 0$ then convergence to θ occurs for all $\tau > 0$, and $\{\xi_n\}$ has extremal index θ. Conversely if $P\{M_n \le u_n(\tau)\} \to e^{-\theta\tau}$ for some $\tau > 0$, $\{\xi_n\}$ has extremal index θ and $P\{\xi_2 \le u_n(\tau) | \xi_1 > u_n(\tau)\} \to \theta$ for all $\tau > 0$. □

The following lemma, giving alternative expressions for θ involves stationarity but does not require any dependence condition.

<u>Lemma 2.5</u> If $n\mu(u_n) \to \upsilon$ the following are equivalent:

(i) $P\{\xi_2 \leq u_n | \xi_1 > u_n\} \to \theta$

(ii) $n(1-F(u_n)) \to \upsilon/\theta$ (i.e. $u_n = u_n(\upsilon/\theta)$)

(iii) $n(1-F_{1,2}(u_n)) \to \upsilon + \upsilon/\theta$. ($F_{1,2}(u_n) = P\{\xi_1 \leq u_n \xi_2 \leq u_n\}$).

Proof: Equivalence of (i) and (ii) is immediate from (2.1). That of (ii) and (iii) follows since

$$\begin{aligned} n\mu(u_n) &= nP\{\xi_1 \leq u_n < \xi_2\} = n(F(u_n) - F_{1,2}(u_n)) \\ &= n((1 - F_{1,2}(u_n)) - (1-F(u_n))) \end{aligned} \qquad \square$$

Write now $\tilde{u}_n(\upsilon)$ to denote a sequence u_n satisfying $n\mu(u_n) \to \upsilon$ and $F(u_n) \to 1$. The next result shows that $n(1-F(\tilde{u}_n(\upsilon))) \to \upsilon/\theta$ when ξ_n has extremal index θ. This will be denoted by the slightly imprecise, but convenient statement "$\tilde{u}_n(\upsilon) = u_n(\upsilon/\theta)$".

<u>Proposition 2.6</u> (i) Suppose $D(\tilde{u}_n(\upsilon))$, $D''(\tilde{u}_n(\upsilon))$ hold for all $\upsilon > 0$, and $\{\xi_n\}$ has extremal index $\theta > 0$. Then $\tilde{u}_n(\upsilon) = u_n(\upsilon/\theta))$ (i.e. $n(1-F(\tilde{u}_n(\upsilon))) \to \upsilon/\theta$ as $n \to \infty$).

(ii) Conversely suppose $D(u_n(\tau))$, $D''(u_n(\tau))$ hold for all $\tau > 0$. If for some τ, θ $u_n(\tau) = \tilde{u}_n(\theta\tau)$, then $u_n(\tau) = \tilde{u}_n(\theta\tau)$ for all $\tau > 0$ and θ is the extremal index of $\{\xi_n\}$.

Proof: To show (i) note that from Proposition 2.1 $P\{M_n \leq \tilde{u}_n(\upsilon)\} \to e^{-\upsilon}$ and hence $P\{\tilde{M}_n \leq \tilde{u}_n(\upsilon)\} \to e^{-\upsilon/\theta}$ ([3], Theorem 3.7.2) where $\tilde{M}_n$ is the maximum of n i.i.d. random variables with the same distribution F as the ξ_i. That is $F^n(\tilde{u}_n(\upsilon)) \to e^{-\upsilon/\theta}$ from which it follows at once that $n(1-F(\tilde{u}_n(\upsilon))) \to \upsilon/\theta$.

(ii) By Lemma 2.5, $P\{\xi_2 \leq u_n(\tau) | \xi_1 > u_n(\tau)\} \to \theta$, hence by Corollary 2.4, this holds for all τ and θ is the extremal index. In particular, $P\{M_n \leq u_n(\tau)\} \to e^{-\theta\tau}$ for all τ. By Proposition 2.1, therefore, $u_n(\tau) = \tilde{u}_n(\theta\tau)$ which completes the proof. □

3. Point Processes of Exceedances and Upcrossings

Let N_n denote the exceedance point process for a level u_n as defined by (1.1), viz. $N_n(B) = \#\{i, 1 \leq i \leq n: i/n \in B, \xi_i > u_n\}$ for $B \subset (0,1]$. Further, write $\tilde{N}_n$ for the "point process of upcrossings", defined on $(0,1]$ as the points $\frac{i}{n}$ such that $\xi_{i-1} \leq u_n < \xi_i$ i.e. $\tilde{N}_n(B) = \#\{i, 1 \leq 1 \leq n: i/n \in B, \xi_{i-1} \leq u_n < \xi_i\}$. It is readily shown that $\tilde{N}_n$ converges in distribution to a Poisson Process under D, D".

<u>Proposition 3.1</u> Suppose $D(u_n)$, $D''(u_n)$ hold for a sequence $u_n = \tilde{u}_n(\nu)$, i.e. $n\mu(u_n) \to \nu$. Then $\tilde{N}_n \xrightarrow{d} N$ where N is a Poisson Process on $(0,1]$ with intensity ν.

Proof: This follows in a standard way from Kallenberg's Theorem ([2] Theorem 4.7):

(i) If $0 < a < b \leq 1$, $\mathcal{E}\tilde{N}([a,b]) \sim n(b-a)\mu(u_n) \to (b-a)\nu = \mathcal{E}N((a,b])$

(ii) $0 \leq P\{\tilde{N}_n((a,b]) = 0\} - P\{M((na,nb]) \leq u_n\} \leq P\{\xi_{[na]+1} > u_n\} = 1 - F(u_n)$

and for disjoint subintervals $(a_i,b_i]$ of $(0,1]$ $1 \leq i \leq k$,

$$0 \leq P\{\tilde{N}_n(\bigcup_1^k (a_i,b_i]) = 0\} - P\{M(\bigcup_1^k (na_i,nb_i]) \leq u_n\}$$
$$\leq k(1-F(u_n)) \to 0$$

and hence by Lemma 1.1 and Corollary 2.2,

$$P\{\tilde{N}_n(\bigcup_1^k(a_i,b_i]) = 0\} = \prod_1^k P\{M((na_i,nb_i]) \le u_n\} + o(1) \to \exp\{-\nu \sum_1^k (b_i-a_i)\}.$$

But this expression is simply $P\{N(\bigcup_1^k(a_i,b_i]) = 0\}$ thus verifying the conditions of Kallenberg's Theorem. □

<u>Corollary 3.2</u> If $D(u_n)$, $D''(u_n)$ hold for a sequence $u_n = u_n(\tau)$ $(n(1-F(u_n)) \to \tau)$ and $\{\xi_n\}$ has extremal index $\theta > 0$, then $\tilde{N}_n \xrightarrow{d} N$ where N is Poisson with intensity $\theta\tau$.

Proof: Since $n(1-F(u_n)) \to \tau$, (2.1) and Corollary 2.4 show that $n\mu(u_n) \to \theta\tau$ so that the proposition applies with $\nu = \theta\tau$. □

The above discussion hinges on the assumption $D''(u_n)$. In that case (as will be seen) each run of consecutive exceedances following an upcrossing may be regarded as a "cluster" of exceedances. If $D''(u_n)$ is not assumed, clusters may consist of groups of "exceedance runs". In general a simple and useful definition of clusters is obtained by choosing k_n to satisfy (1.2) and considering the subintervals $J_i = ((i-1)r_n/n, ir_n/n]$, $1 \le i \le k_n$ of $(0,1]$. Then the exceedances in any interval J_i (i.e. points $\frac{j}{n} \in J_i$ with $\xi_j > u_n$) are regarded as forming a cluster. The "cluster centers" may be defined in an arbitrary way as any point in a J_i containing a cluster - here we use the position of the first event in the cluster. The positions of the cluster centers then form a point process N_n^* for which the following convergence holds (proved similarly to Proposition 3.1).

<u>Proposition 3.3.</u> Suppose $D(u_n)$ holds, where $P\{M_n \le u_n\} \to e^{-\nu}$ for some $\nu > 0$. Then $N_n^* \xrightarrow{d} N$ where N is Poisson with intensity ν. As in Corollary 3.2 if $u_n = u_n(\tau)$ and $\{\xi_n\}$ has extremal index $\theta > 0$ then N has intensity $\theta\tau$. □

In cases where $D''(u_n)$ holds, N_n^* and $\tilde{N}_n$ are asymptotically equivalent as might be expected, in the strong sense of the next result. That is the cluster positions essentially coincide with the upcrossings. It will be seen further (in Proposition 3.5) that cluster sizes then also correspond asymptotically to lengths of exceedance runs, so that clusters and exceedance runs may be identified.

<u>Proposition 3.4</u> Under the conditions of Proposition 3.1 the total variation of the random signed measure $\tilde{N}_n - N_n^*$ satisfies $\mathcal{E}||\tilde{N}_n - N_n^*|| \to 0$ as $n \to \infty$.

Proof: Define a point process N_n' to consist of all points of $\tilde{N}_n$ together with any points $\frac{i}{n}$ for which $\xi_i > u_n$. Then $N_n'(B) \ge \tilde{N}_n(B)$ for each $B \subset (0,1]$, and $||N_n' - \tilde{N}_n|| = N_n'((0,1]) - \tilde{N}_n((0,1])$ so that

$$\mathcal{E}||N_n' - \tilde{N}_n|| \le k_n P\{\xi_1 > u_n\} \to 0 \qquad \text{by assumption.} \tag{3.1}$$

Clearly also $N_n'(B) \ge N_n^*(B)$ and $||N_n' - N_n^*|| = N_n'(0,1] - N_n^*(0,1]$. But $\mathcal{E}N_n'((0,1]) \le \mathcal{E}\tilde{N}_n((0,1]) + k_n(1-F(u_n)) = (n-1)\mu(u_n) + o(1) \to \nu$ and

$$\mathcal{E}N_n^*(0,1] = k_n P\{M_{r_n} > u_n] + o(1) \to \nu$$

by Lemma 1.1 since $P\{M_n \le u_n\} - P^{k_n}\{M_{r_n} \le u_n\} \to 0$ and $P\{M_n \le u_n\} \to e^{-\nu}$. Hence $\mathcal{E}(N_n'(0,1] - N_n^*(0,1]) \to 0$ showing that $E||N_n' - N_n^*|| \to 0$ which combines with (3.1) to give the desired conclusion. □

The discussion of the limiting behavior of the actual exceedance point process N_n requires a dependence restriction of similar type, but somewhat stronger than $D(u_n)$. Such a condition $(\Delta(u_n))$ is used in [1] where it is shown that if $P\{M_n \leq u_n\} \to e^{-\upsilon}$ for some $\upsilon > 0$ then N_n converges in distribution to a Compound Poisson Process provided the cluster size distribution $\pi_n(j)$ converges for each j to $\pi(j)$ a probability distribution on $(1,2,3,\ldots)$. Here the $\pi_n(j)$'s are simply defined to be the distribution of the number of events in a cluster (i.e. in an interval $((i-1)r_n/n,\ ir_n/n]$) given that there is at least one. The Poisson Process underlying this limit has intensity υ and may be regarded as the limiting point process of cluster centers. The distribution for the multiplicity of each event in the Compound Poisson limit is just $\pi(j)$.

It is natural to ask whether the $\pi_n(j)$ may be replaced by the distribution $\pi_n'(j)$ of the length of an exceedance run defined more precisely by

$$\pi_n'(j) = P\{\xi_2 > u_n,\ \xi_3 > u_n, \ldots \xi_{j+1} > u_n,\ \xi_{j+2} \leq u_n | \xi_1 \leq u_n < \xi_2\}$$

That this is the case is shown under $D''(u_n)$ by the following result

<u>Proposition 3.5</u> Suppose $D(u_n)$, $D''(u_n)$ hold where $u_n = \tilde{u}_n(\upsilon)$ for some $\upsilon > 0$. Then $\pi_n(j) - \pi_n'(j) \to 0$ as $n \to \infty$ for each $j=1,2,\ldots$

Proof: It will be more convenient (and clearly equivalent) to show that $Q_n(j)-Q_n'(j) \to 0$ where $Q_n(j) = \sum_{s=j}^{\infty} \pi_n(s)$, $Q_n'(j) = \sum_{s=j}^{\infty} \pi_n'(s)$. Writing J for the interval $(0, r_n/n]$ we have for $j \geq 1$,

$$\begin{aligned} Q_n(j) &= P\{N_n(J) \geq j | N_n(J) > 0\} = P\{N_n(J) \geq j\}/P\{N_n(J) > 0\} \\ &= \frac{k_n}{\upsilon} P\{N_n(J) \geq j\}\ (1+o(1)) \end{aligned}$$

since $P\{N_n(J) > 0\} = P\{M_{r_n} > u_n\} \sim \upsilon/k_n$ (by Lemma 1.1, since $P\{M_n \leq u_n\} \to e^{-\upsilon}$)

so that

$$Q_n(j) = \frac{k_n}{\nu}[P\{\xi_1 > u_n,\ N_n((\tfrac{1}{n}, \tfrac{r_n}{n}]) \geq j-1\}$$
$$+ \sum_{i=1}^{r_n-j+1} P\{\xi_1 \leq u_n, \ldots \xi_{i-1} \leq u_n < \xi_i, N_n((\tfrac{i+1}{n}, \tfrac{r_n}{n}]) \geq j-1\}](1+o(1))$$

Now

$$\frac{k_n}{\nu} P\{\xi_1 > u_n,\ N_n((\tfrac{1}{n}, \tfrac{r_n}{n}]) \geq j-1\} \leq \frac{k_n}{n}(1-F(u_n)) = o(1)$$

and

$$0 \leq P\{\xi_1 \leq u_n, \ldots \xi_{i-1} \leq u_n < \xi_i,\ N_n(\tfrac{i}{n}, \tfrac{r_n}{n}] \geq j-1\}$$
$$- P\{\xi_1 \leq u_n \ldots \xi_{i-1} \leq u_n < \xi_i,\ \xi_{i+1} > u_n, \ldots \xi_{i+j-1} > u_n\}$$
$$\leq P\{\xi_1 > u_n, \bigcup_{j=i+2}^{r_n} (\xi_{j-1} \leq u_n < \xi_j)\}$$
$$\leq \sum_{j=3}^{r_n} P\{\xi_1 > u_n,\ \xi_{j-1} \leq u_n < \xi_j\} = o(1/n)$$

by $D''(u_n)$, so that

$$Q_n(j) = \frac{k_n}{\nu}\left[\sum_{i=1}^{r_n-j+1} P\{\xi_1 \leq u_n, \ldots \xi_{i-1} \leq u_n,\ \xi_i > u_n \ldots \xi_{i+j-1} > u_n\}\right](1+o(1)) + o(1).$$

Also

$$0 \leq P\{\xi_{i-1} \leq u_n,\ \xi_i > u_n, \ldots \xi_{i+j-1} > u_n\} - P\{\xi_1 \leq u_n, \ldots \xi_{i-1} \leq u_n, \xi_i > u_n, \ldots \xi_{i+j-1} > u_n\}$$
$$\leq \sum_{j=3}^{r_n} P\{\xi_1 > u_n,\ \xi_{j-1} \leq u_n < \xi_j\} = o(1/n)$$

so that

$$Q_n(j) = \frac{k_n}{\nu}\Big[\sum_{i=1}^{r_n-j+1} P\{\xi_{i-1} \le u_n,\ \xi_i > u_n, \dots \xi_{i+j-1} > u_n\}\Big]\,(1 + o(1)) + o(1)$$

$$\sim \frac{k_n}{\nu}(r_n - j + 1)\ P\{\xi_2 > u_n, \dots \xi_{j+1} > u_n \,|\, \xi_1 \le u_n < \xi_2\}\ \frac{\nu}{n}(1 + o(1)) + o(1)$$

$$= Q_n'(j)(1 + o(1)) + o(1) = Q_n'(j) + o(1)$$

as required. □

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